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## The *N*-Body Approach to Disk Galaxy Evolution

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**Abstract.** I review recent progress from *N*-body simulations in our understanding of the secular evolution of isolated disk galaxies. I describe some of the recent controversies in the field which have been commonly attributed to numerics. The numerical methods used are widely used in computational astronomy and the problems encountered are therefore of wider interest.

### 1. Introduction

The fragility of disk galaxies suggests that a significant fraction of their history was spent experiencing mild external perturbations. Thus their evolution is likely to have been driven partly by internal processes. Dissipational gas physics probably plays a prominent role in this evolution but the effects of collisionless evolution very near equilibrium are also significant. Collisionless secular evolution is driven largely by non-axisymmetric structures, especially by bars, which represent large departures from axisymmetry, and are thus efficient agents of mass, energy and angular momentum redistribution. Moreover bars occur in over 50% of disk galaxies (Knapen 1999; Eskridge et al. 2000) and simulations starting from the 1970s (Miller et al. 1970; Hohl 1971) have shown that they form readily via global instabilities (Kalnajs 1972; Toomre 1981). Here I review recent progress in understanding the secular evolution of isolated disk galaxies.

### 2. Numerical Methods

At present there is considerable debate on the role numerics play in the evolution of *N*-body simulations. This review therefore treats collisionless isolated galaxy simulations as the necessary prerequisites for confidence in any other *N*-body study of galaxy formation and evolution. The physics of isolated collisionless systems consist only of Newton's law of gravity and Newtonian mechanics, both of which are well understood. Other than initial conditions, the main difficulty lies in computing the gravitational field of a given mass distribution accurately and efficiently. Any gravodynamical code must include at least this much, including those used for studying planet and cosmological structure formation.

Many algorithms have been devised for calculating gravitational fields of isolated galaxies. The simplest is direct particle-particle interactions (e.g. Aarseth 1963), but this scales as  $\mathcal{O}(N^2)$ ,  $N$  being the number of particles, making it impractical for other than special applications. Particle-Mesh (PM/grid) codes (Miller & Prendergast 1968; Miller 1976) solve the potential on a grid by bin-

ning the mass distribution. Several geometries are possible, including cartesian, cylindrical and spherical. Grid codes are very efficient, scaling largely with the number of grid cells,  $N_g$ , as  $\sim \mathcal{O}(N_g)$  (Sellwood 1997). Their main drawbacks are their inflexible geometry and their trade-off between volume and spatial resolution, but hybrid codes can alleviate many of these problems (e.g. Fux 1999; Sellwood 2003). Adaptive Mesh Refinement (AMR) codes (e.g. Bryan & Norman 1995; Kravtsov et al. 1997), on the other hand, let the grid evolve dynamically to increase the resolution in dense regions. Tree codes (e.g. Barnes & Hut 1986; Hernquist 1987) instead group particles by location, computing direct forces for nearby particles and a few multipoles for the more distant ones. These codes can be vectorized efficiently and scale as  $\mathcal{O}(N \log N)$  (Dubinski 1996; Stadel 2001; Springel et al. 2001) while Dehnen (2000) presented an  $\mathcal{O}(N)$  extension. Tree codes have been widely implemented and several are publicly available. Self-Consistent Field (SCF) codes (e.g. Clutton-Brock 1972; Hernquist & Ostriker 1992; Earn & Sellwood 1995) expand the density in a set of orthogonal basis functions, from which the potential can then be computed. SCF codes scale as  $\mathcal{O}(N)$  and do not require softening because truncating the expansion is enough to suppress small scale noise. However they require that the basis set be chosen with care or that a large number of basis functions be included (which re-introduces small scale noise). Weinberg (1996) solves this problem by making the basis set adaptive.

## 2.1. Code testing

The first requirement of any  $N$ -body code used to study galaxy evolution is that it be collisionless. Relaxation rates of conserved quantities are useful diagnostics for this purpose (e.g. Hohl 1973; Hernquist & Barnes 1990; Hernquist & Ostriker 1992; Valenzuela & Klypin 2003).

Testing the gravodynamical part of a code requires more effort. Comparison with the limited number of exact analytic results known provide stringent code tests; such systems (in 2-D) include the Kalnajs disk (Kalnajs 1972), the Mestel disk (Zang 1976), the isochrone disk (Kalnajs 1978) and the power-law disks (Evans & Read 1998a,b). Several code tests using these predictions have been carried out. Earn & Sellwood (1995) compared the eigenfrequencies of instabilities in isochrone disks with simulations using an SCF and a PM method. They were able to reproduce the predicted values to within 5% with just  $15K$  particles using the SCF method whereas softening caused the PM code to never quite converge to the predicted value. Meanwhile, Sellwood & Evans (2001) presented  $N$ -body examples of the power-law disks including reproducing a challenging case of a perfectly stable disk. They reported that aliasing rendered SCF codes ill-suited to these disks. Instabilities of spherical systems have also been used for code testing. For example Weinberg (1996) tested his adaptive SCF code on the instabilities of spherical generalized polytropes investigated by Barnes et al. (1986) and found generally good agreement. Analytic solutions of non-equilibrium evolution, which include the solution of 1-D plane wave collapse (Zel'Dovich 1970) and of spherical infall in an expanding universe (Fillmore & Goldreich 1984; Bertschinger 1985) have also be used to test  $N$ -body codes (Kravtsov et al. 1997; Davé et al. 1997).

Different codes can be tested also by direct comparison. Inagaki et al. (1984) compared bar formation between an *N*-body model and a direct numerical integration of the CBE and Poisson equation (Nishida et al. 1981; Watanabe et al. 1981). The two simulations matched each other to better than 2% in bar amplitude well into the non-linear regime and eventual discrepancies were due to the inability of the CBE code to handle large gradients in the distribution function (DF).

### 3. Cusp evolution

A prediction of cold dark matter (CDM) cosmology is that dark matter halos are cusped, with densities  $\rho \sim r^{-\beta}$  at small radii and  $1 \leq \beta \leq 1.5$  (Navarro et al. 1997; Moore et al. 1998; Jing & Suto 2000; Power et al. 2003). Several arguments have been advanced against the presence of such cusps in real galaxies, including detailed rotation curve fits of dwarf and low surface brightness galaxies (Blais-Ouellette et al. 2001; de Blok et al. 2001; Matthews & Gallagher 2002), bar gas flows (Weiner et al. 2001; Pérez et al. 2004) and pattern speeds (Debattista & Sellwood 1998, 2000, see §4. below) and the microlensing optical depth and gas dynamics in the Milky Way (Binney et al. 2000; Bissantz et al. 2003).

Several ways to erase cusps have been considered including new dark matter physics (Spergel & Steinhardt 2000; Peebles 2000; Goodman 2000; Kaplinghat et al. 2000; Cen 2001), feedback from star formation (Navarro et al. 1996; Gnedin & Zhao 2002), or an initial power spectrum with decreased power on small scales (Hogan & Dalcanton 2000; Colín et al. 2000; Bode et al. 2001). Binney et al. (2001) suggested that bars in young galaxies were able to torque up cusps and expel them, while Hernquist & Weinberg (1992) had found that such torques can reduce the density of a spheroid by a factor of  $\sim 100$  out to  $\sim 0.3a_B$  (where  $a_B$  is the bar's semi-major axis). Weinberg & Katz (2002, hereafter WK02), following Hernquist & Weinberg (1992), presented perturbation theory calculations and simulations of imposed rigidly-rotating non-slowing (IRRNS) bars to argue that cusps are erased in a few bar rotations ( $\sim 10^8$  years). Their bars were large ( $\sim 10$  kpc), but they argued for a scenario in which large primordial bars destroy cusps while the current generation of smaller bars formed later. Large reductions of dark halo densities were not seen in self-consistent cuspy halo simulations (e.g. O'Neill & Dubinski 2003; Valenzuela & Klypin 2003); WK02 suggested that this was because the resonant dynamics responsible for cusp removal are very sensitive to numerical noise. They found that they needed  $N > 10^6$  for their SCF code and argued that even larger  $N$  would be needed for grid, tree or direct codes.

The scenario of WK02 makes three main claims: (1) that bars cause a decrease in cusp density and can destroy cusps if sufficiently large and strong (2) that such bars formed via interactions at high redshift and (3) that  $N$  needs to be large in order that the relevant phase space is adequately covered and that orbit diffusion does not destroy the resonant dynamics causing cusp destruction.

### 3.1. When do bars destroy cusps?

Sellwood (2003, hereafter S03) presented simulations of the same IRRNS bar as was used by WK02 and found cusp destruction occurred in a runaway process. The inclusion of the  $l = 1$  spherical harmonic terms in the potential had a large effect on the evolution; in their absence, cusp removal took  $\sim 5 - 6$  times longer than when they were present. For the same IRRNS bar, McMillan & Dehnen (2005, hereafter MD05) found that the cusp moves off-center by as much as 30% of  $a_B$ . This centering instability gives the appearance that the cusp has been erased when halo density is measured by spherically averaging about the origin which, they argued, caused WK02 to over-estimate a bar's ability to erase a cusp. Evidence for lopsidedness in the simulation of WK02 can be inferred from its asymmetric bar-induced halo wake (their figure 2). When MD05 suppressed this purely numerical instability they still found that the cusp is destroyed, although after a much longer time. The possibility that an offset cusp is being confused for cusp destruction was investigated by Holley-Bockelmann et al. (2005, hereafter HB05). Their self-consistent simulations formed bars in a disk of particles, thus damping any centering instability. While confirming that some of the evolution seen by WK02 was due to the centering instability, they found that when odd  $l$  terms were excluded in their simulations that the damage to the cusp was not significantly diminished.

Thus these idealized bars at least are able to destroy cusps. What about more realistic bars? S03 showed that when, instead of assuming a fixed pattern speed,  $\Omega_p$ , he allowed it to decrease such that total angular momentum is conserved assuming a constant moment of inertia (an IRRS bar), that the cusp was damaged significantly less. This happens because an IRRNS bar must transfer more angular momentum to the halo than it can plausibly have in order to destroy the cusp. Shorter, more realistically sized bars were also ineffective bar destroyers. Similarly MD05 found that an IRRS bar is unable to destroy a cusp in a Hubble time. S03 also presented self-consistent simulations in which a disk of particles was grown inside a halo. In these cases, besides forming smaller bars which slow, cusp erasure was also inhibited by the increase in halo density as the disk grew and again as the central density of the disk increased because of angular momentum transport outwards. HB05 instead found that the large relative angular momentum gained by the inner halo in their self-consistent simulations flattened the cusp, but only out to 900 pc scaled to the Milky Way.

### 3.2. Do bars get as large as needed?

Thus all self-consistent simulations find that bars of sizes typical of those observed are unable to remove cusps on scales of several kpc. Do bars ever get to be large enough to do so? Bars that form via disk instabilities (Toomre 1981) generally extend to roughly the radius of the rotation curve turnover. However, externally triggered bars may be substantially larger. HB05 showed an example of such a bar; scaled to the Milky Way, external triggering produced a bar of  $a_B = 12$  kpc, as opposed to  $\sim 4.5$  kpc via the bar instability. Whether the required triggering actually occurs is a question for hierarchical simulations while direct observations at high redshift should establish whether bars get to be as large as 10 kpc.

### 3.3. Is the evolution in simulations compromised by too small $N$ ?

Weinberg & Katz (2005) provided several reasons for the need of large  $N$ . They focused especially on the inner Lindblad resonance (ILR) which always extends down to the cusp. One possible problem they identified is two-body scattering which leads to particles executing a random walk and therefore lingering in resonance for significantly less than they would otherwise. In the terminology of Tremaine & Weinberg (1984), scattering causes particles to traverse the ILR in the fast rather than the slow regime. Since resonant torques scale as  $m_P^2$  in the fast regime and as  $\sqrt{m_P}$  in the slow regime (where  $m_P$  is the fractional mass of the perturber) two-body scattering causes the friction to be substantially reduced. For a typical bar, they argued that  $N > 10^8$  within the virial radius is required. A second limiting factor they identified is phase space coverage. Whether a resonant particle gains or loses angular momentum depends on its phase. If the phase space of the resonance is not sufficiently sampled by particles then the correct ensemble average is not attained.

HB05 found no difference between their tidally perturbed simulations with  $N = 5.5M$  and  $N = 11M$ , while  $N = 1.1M$  resulted in a weaker bar and less friction than in the other two cases. This, they argued, proved that the  $N = 1.1M$  simulation suffered from too much orbital diffusion to follow resonances correctly. However, Sellwood (2005) argued plausibly that this  $N$  dependence was due to the bar being weaker, and not because friction depends on  $N$ . The weaker bar in the  $N = 1.1M$  simulation was caused by the unavoidably larger  $m = 2$  seed amplitude in the initial disk which, once swing-amplified, interfered destructively with the distortion induced by the externally applied tidal field.

The simulations of S03 (using both a PM and an SCF code) and MD05 (using a tree code) are instructive because they studied the same physical system: a Hernquist (1990) halo with a IRRNS bar of  $a_B = 0.7r_s$  ( $r_s$  being the halo scale radius). Both studies found that the onset of the runaway cusp destruction depended on  $N$  and occurs earlier with *decreasing*  $N$ . S03 found good agreement between evolution using the PM code and the SCF code; additionally he showed that using only  $l = 0$  and 2 terms was sufficient for the evolution, with little change when larger even values of  $l$  were included. On the other hand, MD05 forced symmetry about the origin to suppress the centering instability. Thus the two studies were very similar; but while their results are in qualitative agreement, there are surprising quantitative differences. S03 found that the onset of the runaway appears to be converging by  $N = 3M$  at  $\sim 130$  bar rotations (his figure 2b) but for the same  $N$  MD05 saw no evidence for convergence with runaway at  $\sim 210$  bar rotations (their figure 5b). Yet both agreed that the radius containing 1% of the total mass is driven from  $\sim 0.1r_s$  to  $\sim 0.5r_s$ . Unless the difference is due to the initial conditions in some worrisome way, noise would seem to be responsible for these differences. A more careful comparison between these different simulations seems particularly worthwhile.

Sellwood & Debattista (2005) report an entirely different test of whether scattering overwhelms bar evolution. They presented simulations (using a PM code) in which they perturbed the system into a metastable state in which  $\Omega_p$  was nearly constant (more on this in §4.). Their metastable state resulted because  $\Omega_p$  was driven up such that the principal resonances are trapped in

shallow local minima of the DF. Systems persisted in this metastable state for  $\sim 5$  Gyr, which would not have been possible if orbit scattering had been strong.

Unlike two-body scattering, phase space coverage can be investigated by means of test particle simulations, where the self-gravitating halo response is not included. Using such experiments, Sellwood (2005) found that  $N \sim 1M$  was sufficient for the  $\Omega_p$  evolution of a low mass bar  $M_{\text{bar}} = 0.005M_{\text{halo}}$  to converge, and a factor of  $\sim 100$  less particles were needed for  $M_{\text{bar}} = 0.02M_{\text{halo}}$ . When self-gravity was introduced (thus adding scattering),  $1M$  particles were needed for the  $M_{\text{bar}} = 0.02M_{\text{halo}}$  case. As in S03, he argued that WK02 needed large  $N$  because of the extremely difficult nature of their experiments with a fixed-amplitude bar rotating at a fixed  $\Omega_p$ . When  $\Omega_p$  is a function of time, the resonances are broadened to orbits near resonance, which makes phase space coverage a less stringent constraint on  $N$ . Weak bars in particular may be prone to such difficulties, but it is strong bars which are most interesting.

In summary, there may be some evidence that noise is compromising some if not all of the evolution in some cases, in ways not yet fully understood. But it is clear that only quite large bars can erase cusps to scales larger than 1 kpc.

#### 4. Evolution of pattern speeds

The pattern speed of a bar is usually parametrized by the ratio  $\mathcal{R} \equiv D_L/a_B$ , where  $D_L$  is the corotation radius, at which the gravitational and centrifugal forces cancel. A self-consistent bar must have  $\mathcal{R} \geq 1$  (Contopoulos 1980). A bar is termed fast when  $1.0 \leq \mathcal{R} \leq 1.4$  and slow otherwise. This definition does not distinguish fast from slow in terms of  $\Omega_p$  alone: a bar in a galaxy with rotation velocity of  $200 \text{ km s}^{-1}$  is slow at  $\Omega_p = 100 \text{ km s}^{-1} \text{ kpc}^{-1}$  if  $a_B = 1 \text{ kpc}$ , but fast at  $\Omega_p = 20 \text{ km s}^{-1} \text{ kpc}^{-1}$  if  $a_B = 10 \text{ kpc}$ . Observational evidence points to fast bars, both in early-type barred galaxies (Merrifield & Kuijen 1995; Gerssen et al. 1999; Debattista et al. 2002; Aguerri et al. 2003; Gerssen et al. 2003; Debattista & Williams 2004) and in late-types (Lindblad et al. 1996; Lindblad & Kristen 1996; Weiner et al. 2001; Pérez et al. 2004).

Tremaine & Weinberg (1984) developed the perturbation theory of dynamical friction for perturbers in spheroidal systems, showing that friction arises near resonances, when  $m\Omega_p = k\Omega_r + l\Omega_\phi$ , where  $\Omega_r$  and  $\Omega_\phi$  are the radial and angular frequencies, respectively (see Weinberg 2005, for a time-dependent treatment). Weinberg (1985) applied this theory to a bar rotating in a dark halo and found that the bar is braked such that  $D_L \gg a_B$  unless (1) angular momentum is added to the bar, (2) the bar is weak or (3) the halo has low mass.

The transfer of angular momentum from disk to spheroid (bulge or halo) was reported in several early simulations (Sellwood 1980; Little & Carlberg 1991; Hernquist & Weinberg 1992). Fully self-consistent, 3-D simulations of bar-unstable disks embedded in dark halos were presented by Debattista & Sellwood (1998, 2000, hereafter DS00) who found that disks needed to be the dominant mass component if the bars which form were to remain at  $\mathcal{R} < 1.4$ . Bar slowdown has also been found in other simulations since then (O'Neill & Dubinski 2003; Holley-Bockelmann et al. 2005; Berentzen et al. 2005; Martinez-Valpuesta et al. 2005). O'Neill & Dubinski (2003) included a comparison with DS00 and found generally good agreement.

The constraint of DS00 severely limits CDM cusps to be present in dark halos. Valenzuela & Klypin (2003, hereafter VK03) presented simulations of a CDM Milky Way, with halo concentration  $c \simeq 15$ . These simulations were evolved on an AMR code with a maximum refinement corresponding to a resolution of 20-40 pc. They found that the bar which formed remained fast for  $\sim 4-5$  Gyr, thereafter slowing to  $\mathcal{R} = 1.7$ . They concluded that the slow bars in the simulations of DS00 were an artifact of low resolution. They also argued that their bars remained modest in length (1-2 scale-lengths) but became too long in lower resolution simulations, which they concluded again proved the limitations of low resolution simulations. Sellwood & Debattista (2005) repeated one of the simulations of VK03 using the same initial conditions but evolved on a hybrid grid code (Sellwood 2003) with a fixed softening comparable to that of VK03. They found instead that the bar reached  $\mathcal{R} = 2$  within 4 Gyr. They accounted for the result of VK03 by noting that as the bar formed, the central density increased (see §5.). Thus in an AMR code there is a tendency for the spatial resolution to increase; the disk in the model of VK03 being rather thin, this led to enhanced forces and a corresponding artificial increase in  $\Omega_p$ . Once this happened, the bar found itself with its principal resonances at local phase space minima previously generated by the forming bar. Although angular momentum is exchanged at resonances, the sign of the torque depends on the phase space gradient of the DF at the resonances; in the absence of a gradient no friction is possible. Thus the numerics induced a metastable state. This state proved fragile to realistic perturbations and is not likely to last long in nature, but in the quieter environment of an isolated *N*-body system it can persist for many Gyrs, enough to fully account for the behavior found by VK03.

In a series of papers, Athanassoula presented several arguments against the conclusion of DS00. In Athanassoula (2003) she compared the evolution of two halo-dominated systems, MQ2 and MHH2, with nearly identical disk and halo rotation curves. However, their velocity dispersions were different, being larger in MHH2 because its halo extended to larger radii. As a result  $\Omega_p$  decreased significantly in MQ2 but hardly at all in MHH2. She concluded that this “argues against a link between relative halo content and bar slowdown...” This claim however is contradicted by her own simulations. The confusion arises from her relying on the change in  $\Omega_p$  to constrain the halo. Not only is this unobservable, but if we compute  $\mathcal{R}$  for her models, we find  $1.4 < \mathcal{R} \leq 1.7$  for run MQ2 and  $\mathcal{R} \simeq 3$  for MHH2, *i.e.* both these halo-dominated systems are slower than observed. Far from being in disagreement with the results of DS00, her simulations support them. She also showed that weak bars are less able to drive angular momentum exchanges and may remain fast; since observational measurements have only been obtained for strong bars, this is not worrisome.

Athanassoula & Misiriotis (2002) argued that bar lengths are difficult to measure in simulations making a comparison with corotation difficult. Bar lengths are certainly not always easy to measure, but it is still possible to define values which straddle the real value; for example DS00 had cases in which the uncertainty in  $a_B$  was as large as  $\Delta a_B / \bar{a}_B \sim 0.3$ , not substantially different than in observations (Debattista 2003). But these are not Gaussian errors and the probability of  $a_B$  falling outside the given range is practically zero. Moreover, the same problem afflicts observations; therefore measurement uncertainties in  $a_B$  are a nuisance but not a repudiation of the constraint.

Athanassoula (2002) showed that loss of angular momentum from the bar leads to a growing bar, as first suggested by DS00. VK03 pointed out that bars become excessively long in the presence of strong friction. Even though angular momentum redistribution leads to larger disk scale-lengths, bars extending  $\gtrsim 10$  kpc are not common (Erwin 2005). Strong bar growth does not seem to have occurred through the history of the current generation of bars and presumably, neither has strong friction.

Thus to date no well-resolved simulation has provided a valid counter-example to the claim by Debattista & Sellwood (1998) that dense halos cannot support fast strong bars. If anything, recent simulations have lent support to it.

## 5. The secular evolution of disk densities

The excellent recent review by Kormendy & Kennicutt (2004) presents the observational evidence for pseudo-bulge formation via secular evolution and discusses some of the older  $N$ -body results in that field. Here I review recent developments not covered by those authors.

### 5.1. Disk Profile Evolution

Bar formation is accompanied by a rearrangement of disk material as first shown by Hohl (1971). Generally a nearly double-exponential profile develops, with a smaller central scale-length than the initial and a larger one further out. In this respect these profiles resemble bulge+disk profiles and comparisons with observations show that these profiles are reasonable approximations to observed profiles (Debattista et al. 2004; Avila-Reese et al. 2005). Debattista et al. (2005b) showed that the degree by which the profile changes depends on the initial disk temperature  $Q$ . In hot disks ( $Q \sim 2$ ) little angular momentum needs to be shed and the azimuthally-averaged density profile is practically unchanged. Thus the distribution of disk scale-lengths depends not only on the initial angular momentum of the baryons (and presumably that of the dark halo) but also the disk temperature. Angular momentum redistribution continues also after the bar forms, especially to the halo. This leads to a further increase in the central density of the disk even when the evolution is collisionless. This may be sufficient to render an initially halo-dominated system into one dominated by baryons in the inner parts (Debattista & Sellwood 2000; Valenzuela & Klypin 2003).

The angular momentum lost by the bar as it forms may be transported out to large radii via a resonant coupling between the bar and spirals (Debattista et al. 2005b) of the kind found by Masset & Tagger (1997) and Rautiainen & Salo (1999). Debattista et al. (2005b) show that this transport leads to breaks in the density distribution which, when viewed edge-on, are indistinguishable from those observed in real galaxies (Pohlen 2002).

### 5.2. Vertical Evolution: Peanut-Shaped Bulges

$N$ -body simulations of the vertical evolution of disk galaxies have concentrated on box- or peanut- (B/P-) shaped bulges which are present in some 45% of edge-on galaxies (Lüticke et al. 2000). Simulations have shown that these form

via the secular evolution of bars (Combes & Sanders 1981), either through resonant scattering or through bending (aka buckling) instabilities (Pfenniger 1984; Combes et al. 1990; Pfenniger & Friedli 1991; Raha et al. 1991). Although it is often thought that a peanut requires that a bulge was built by secular processes, simulations show that peanuts can also form when the initial conditions include a bulge, as would happen if bulges form through mergers at high redshift (Athanassoula & Misiriotis 2002; Debattista et al. 2005a).

Observations seeking to establish the connection between B/P-shaped bulges and bars (Kuijken & Merrifield 1995; Merrifield & Kuijken 1999; Bureau & Freeman 1999; Chung & Bureau 2004), by looking for evidence of bars in edge-on B/P-bulged systems, have benefited from comparisons with the edge-on stellar velocity distributions of *N*-body bars (Bureau & Athanassoula 1999, 2005). Bureau & Athanassoula (2005) characterized the signature of an edge-on bar as having (1) a Freeman type II profile, (2) a rotation curve with a local maximum interior to its flat part (3) a broad velocity dispersion profile with a plateau at intermediate radii (4) a correlation between velocity and the third-order Gauss-Hermite moment  $h_3$  (Gerhard 1993; van der Marel & Franx 1993). A diagnostic of B/P-shaped bulges in face-on galaxies was developed, and tested on high mass and force resolution simulations, by Debattista et al. (2005a). Vertical velocity dispersions constitute a poor diagnostic because they depend on the local surface density. Instead, their diagnostic is based on the fact that peanut shapes are associated with a flat density distribution in the vertical direction. The kinematic signature corresponding to such a distribution is a minimum in the fourth-order Gauss-Hermite moment  $h_4$ .

The buckling instability itself has also been studied with simulations. Debattista et al. (2005b) showed that an otherwise vertically-stable bar is destabilized when it slows and grows. Moreover, the instability can occur more than once for a given bar (Martinez-Valpuesta et al. 2005). Finally, after Raha et al. (1991) showed that buckling weakens bars, it has often been assumed that bars are destroyed by buckling. Debattista et al. (2005b) presented a series of high force and mass resolution simulations demonstrating that this is not the case.

### 5.3. Spirals

There is broad agreement that spirals constitute density waves (Lin & Shu 1964). At least three different dynamical mechanisms have been proposed for exciting them: swing-amplification (Toomre 1981), long-lived modes (Bertin et al. 1989a,b) and recurrent instabilities seeded by features in the angular momentum distribution (Sellwood & Lin 1989; Sellwood & Kahn 1991; Sellwood 2000). All three are still viable and not much new *N*-body results have been obtained in recent years, but Sellwood & Binney (2002) used simulations to demonstrate that spirals cause a considerable radial shuffling of mass at nearly fixed angular momentum distribution. This happens at corotation and is not accompanied by substantial heating — stars on nearly circular orbits can be scattered onto other nearly circular orbits. For example, they estimate that a star born at the solar radius can be scattered nearly uniformly within  $\Delta R = \pm 4$  kpc. Thus the idea of a Galactic habitable zone becomes somewhat suspect, as does the need for infall to maintain the metallicity distribution observed at the solar circle.

## 6. Bar destruction by CMCs

Central massive concentrations (CMCs), whether supermassive black holes (hard CMCs) or gas condensations several hundred parsecs in size (soft CMCs), could destroy bars. Early studies of this phenomenon were inspired by the similar work for slowly-rotating triaxial elliptical galaxies (e.g. Gerhard & Binney 1985), where the loss of triaxiality results from the destruction of box orbits by scattering off the CMC. Such scattering may not be efficient in bars, since the main bar-supporting orbits are loops which avoid the center. Hasan & Norman (1990) and Hasan et al. (1993) argued that, when a CMC in a barred galaxy grew sufficiently massive, it quickly destroys bar-supporting orbits by driving an ILR, around which orbits are unstable, to large radii. The more centrally concentrated the CMC grew, the further out was the ILR and therefore the more disruptive it was. How massive the CMC needed to be required  $N$ -body simulations to establish. Friedli & Benz (1993) modeled gas and stars with PM+SPH simulations and found that gas inflows destroyed bars when 2% of the baryonic mass ended up in a hard CMC. Norman et al. (1996) pursued collisionless 2-D and 3-D simulations, with CMCs grown by slowly contracting a massive component. They needed a CMC of mass 5% of disk mass,  $M_d$ , to destroy bars. Then bar destruction was rapid and led to a bulge-like spheroid.

Shen & Sellwood (2004, hereafter SS04) presented a series of high-quality simulations including high mass, force and time resolution and found that bars are more robust than previously thought. They varied the growth rates, compactness and mass of the CMCs and considered both weak and strong bars within which they grew CMCs at fixed compactness. A rigid halo with a large core radius was also included. They obtained a fast decay while a CMC was growing followed by a more gradual decay once the CMC reached its full mass. The time over which the bar was grown proved unimportant, with only the final mass and compactness mattering. Hard CMCs cause more damage than soft ones, needing  $4 - 5\% M_d$  and  $> 10\% M_d$ , respectively, to destroy bars. Their tests showed that time steps need to be as small as  $10^{-4}$  of a dynamical time in order that more rapid but incorrect bar destruction is avoided. They interpreted the two-phase bar destruction as scattering of low energy bar-supporting  $x_1$  orbits during the CMC growth phase and continued gradual global structural adjustment, which further destroyed high energy  $x_1$  orbits, thereafter. They predicted that massive halos, which lead to bar growth (Debattista & Sellwood 2000; Athanassoula 2003) render bars even more difficult to destroy; this prediction, as well as the two-phase bar weakening, was confirmed in live-halo simulations by Athanassoula et al. (2005).

Bournaud & Combes (2002) and Bournaud et al. (2005) have argued for a radically different picture. They simulated gas accretion and noted that bars were destroyed and reformed 3 or 4 times over a Hubble time. The amount of gas required to destroy bars is not, however, wholly consistent in these simulations: a system with  $\sim 7\%$  total gas mass fraction lost its bar within 2 Gyr in Bournaud et al. (2005) whereas a system with three times more gas maintained its bar in Bournaud & Combes (2002). Possibly star-formation, included in the later simulations, somehow quenched the infall onto the center. SS04 hinted that the timestep used by Bournaud & Combes (2002) was too large but Bournaud et al. (2005) reported that using a timestep  $0.125 \times$  their standard

value (and close to that advocated by SS04) still led to recurrent bar destruction. Bournaud et al. (2005) argued that their results are correct and that other studies had erred in mimicking gas accretion by simply growing a massive object because this neglected the important effects of angular momentum transport from gas to the bar. This is, perhaps, consistent with the simulations of Berentzen et al. (1998) (who were able to destroy bars by gas inflow leading to a CMC of mass fraction just 1.6%) and the earlier ones of Friedli & Benz (1993). On the other hand, the fully live simulations of Debattista et al. (2005b) only destroyed bars when soft CMCs reached  $\sim 20\%M_d$ , in good agreement with SS04. One possibly important difference is that the simulations of Bournaud et al. (2005) included a rigid halo, which prevents it from accepting angular momentum from the bar, while Debattista et al. (2005b) had live halos.

Hozumi & Hernquist (2005), using a 2-D SCF code, also concluded that hard CMCs can destroy bars with smaller masses,  $0.5\%M_d$ . SS04 (see also Sellwood 2002) speculated that low order SCF expansions may not be able to simultaneously maintain a system axisymmetric near the center and non-axisymmetric further out.

## 7. Multiple patterns

An emerging field in the past few years has been galaxies with multiple patterns. These are challenging to study because traditional tools such as surfaces-of-section are no longer viable. *N*-body simulations, therefore, are vital for studying systems such as bars in triaxial halos and bars within bars.

### 7.1. Bars in triaxial halos

CDM predicts that dark matter halos are triaxial (Barnes & Efstathiou 1987; Frenk et al. 1988; Dubinski & Carlberg 1991; Jing & Suto 2002). The condensation of baryons inside triaxial halos drives them to rounder shapes, but systems do not generally become wholly axisymmetric (Dubinski 1994; Kazantzidis et al. 2004). Using rigidly-rotating bars, El-Zant & Shlosman (2002) computed the Lyapunov exponents of orbits and showed that chaos quickly dominates the evolution when halos are triaxial and cuspy. *N*-body simulations by Berentzen et al. (2005) indeed show that bars are destroyed in cases where the triaxiality in the potential is as small as  $c/a \sim 0.9$  and the halo is cuspy. One way in which this fate can be avoided is for the bar to alter the shape of the inner halo. At present it is not clear which systems can accomplish this and which cannot. A better understanding of when bars are destroyed in triaxial halos could possibly lead to an important new constraint on the shapes and profiles of dark matter halos.

### 7.2. Bars within bars

While observations of largely gas-free early-type galaxies find an abundance of nuclear bars within large scale bars (Erwin & Sparke (2002) found them in  $\sim 30\%$  of barred S0-Sa galaxies), simulating them has proved difficult. Moreover, it is only recently that direct observational evidence for kinematically decoupled primary and nuclear bars in one system has been obtained (Corsini et al. 2003). Therefore their dynamics have been poorly understood, in spite of the fact that

they have been postulated to be an important mechanism for driving gas to small radii to feed supermassive black holes (Shlosman et al. 1989).

Most numerical studies have required gas to form secondary bars (Friedli & Martinet 1993; Shlosman & Heller 2002; Englmaier & Shlosman 2004), but their presence in gas-poor early-type galaxies suggests that gas is not the main ingredient for forming secondary bars. Stellar counter-rotation can lead to counter-rotating bars (Sellwood & Merritt 1994; Friedli 1996). Rautiainen et al. (2002) were the first to succeed in producing collisionless  $N$ -body simulations with both bars rotating in the same sense. Their secondary bars, which were vaguely spiral-like and possibly hollow, rotated faster than the primary bars and survived for several Gyrs. Debattista & Shen (2006, in preparation, see also Shen & Debattista in these proceedings) present further examples and explore the mutual evolution of the two bars. Now that  $N$ -body simulations can achieve the high force and mass resolution needed to form self-consistent nested bar systems it is hoped that progress in understanding these systems will be more rapid than in the 30 years since their discovery (de Vaucouleurs 1975).

## 8. Desiderata for the future

The collisionless simulation of isolated galaxies is an endeavor over thirty years old. The subject is still rich, with several open problems, and continues to be very active. Algorithmically, if not conceptually, it is the simplest problem that can be considered. Various gravity solvers for its study are available (which are used also in other areas of astronomy). Despite much progress, the degree of disagreement in the field, as described above, is surprising. The way to progress from this point is to compare directly the results of different codes with each other, as has been done in other fields (e.g. Kang et al. 1994; Frenk et al. 1999). Unfortunately the number of such tests in galaxy evolution has been small. Therefore a detailed comparison between many different codes would be very valuable at this time. Ideally this would involve several implementations of the same code type to establish behaviors in the different types. The actual tests to be performed should include systems in which an analytic result is known (useful to establish values of numerical parameters which are optimal) as well as systems for which the result is not known in advance. Furthermore, the  $N$ -body tests of the type recently proposed by Weinberg & Katz (2005) provide a challenging and useful basis for assessing the effect of noise on simulations. Such a comparison will give the community greater confidence that we are able to model correctly the most basic level of galaxy evolution.

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